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1987 J. Phys. A: Math. Gen. 20 L539

(http://iopscience.iop.org/0305-4470/20/8/010)

J. Phys. A: Math. Gen. **20** (1987) L539-L542. Printed in the UK

LETTER TO THE EDITOR

All direct sum representations of the Temperley–Lieb algebra

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Received 4 February 1987

Abstract. We show how to write down the irreducible representations of the Temperley-Lieb algebra. We show that these representations satisfy Jones' trace condition in the infinite limit.

There has recently been much interest in the Temperley-Lieb algebra (Temperley and Lieb 1971) for the operators $\{U_i, i = 1, 2n-1\}$

$$\begin{aligned} &U_iU_i = q^{1/2}U_i \\ &U_iU_{i\pm1}U_i = U_i \\ &U_iU_j = U_jU_i \qquad |i-j|>1. \end{aligned} \tag{1}$$

This is because the transfer matrices for various statistical mechanical models may be written in terms of different representations of these operators. This in turn establishes an equivalence between such models, most notably the $q = 4\cos^2(\pi/r)$-state Potts model (see Baxter 1982) and the apparently conformally invariant critical $(r-1)$-state eight-vertex solid-on-solid model (Andrews *et al* 1984, Kuniba *et al* 1986). The operators are also of interest in the theory of von Neumann algebras (Jones 1983) and in knot theory (Jones 1985).

Temperley (1986) has shown how to write down a set of representations of operators satisfying the relations (1) which are the irreducible representations at $q = 4$, but which are not in general irreducible (see Martin 1986a). In the present letter we show how to write down the irreducible representations for general q, and in particular we discuss the dimension of these representations.

Consider a sequence $\{S_j\}$ of $2n+1$ positive integers. The first integer, $S_{1,j}$, is one and the last, $S_{2n+1,j}$, is $2m+1$ ($m = 0, 1, 2, \ldots, n$). Each entry differs from adjacent entries in the sequence by one, and each entry is less than $\tilde{r}$ where $\tilde{r}$ is the numerator of r (taking r to be a rational in its lowest terms, so $\tilde{r}$ is infinite for an irrational). For example, the allowed sequences for $n = 2$, $r = 5$, $m = 1$ are

1 2 1 2 3
1 2 3 2 3
1 2 3 4 3.

The set of such sequences forms a basis for the mth representation, the matrix element $(U_{i-1})_{j,k}$ being zero unless the jth and kth sequences are identical in all but (possibly) the ith entry and $S_{i-1,j} = S_{i+1,j}$ ($S_{i,j}$ is the ith entry in the jth sequence), whereupon the matrix element is

$$\frac{\sin(S_{i,j}\pi/r)}{\sin(S_{i+1,j}\pi/r)} \qquad \text{if } S_{i,j} = S_{i,k}$$

or

$$\frac{[\sin(S_{i,j}\pi/r)\sin(S_{i,k}\pi/r)]^{1/2}}{\sin(S_{i+1,j}\pi/r)} \qquad \text{otherwise.} \tag{2}$$

In order to see that these matrices obey the relations (1) (with $q = 4\cos^2(\pi/r)$) note that the sequences are all allowed configurations of an n-step diagonal layer of the $(\tilde{r}-1)$-state model (provided no sequence entry exceeds $\tilde{r}-1$, see also equation (2)). Note further that allowed configurations of adjacent layers are in the same set of sequences. It therefore follows from the star-triangle relation for this model (see Kuniba *et al* 1986, Baxter 1982) that the matrices obey the relations (1). For example, the star-triangle relation implies

$$(I + x(u)U_j)(I + x(v)U_{j+1})(I + x(v-u)U_j) = (I + x(v-u)U_{j+1})(I + x(v)U_j)(I + x(u)U_{j+1})$$

for any v, u, where

$$x(u) = \frac{\sin u}{\sin(\pi/r - u)}.$$

Then putting $v = 0$ we find

$$U_j^2 - q^{1/2}U_j = k$$

where the operator k is the same for any j. But from equation (2) we find that $k = 0$ whenever $j = 1$, thus giving the first of the relations (1).

It is easy to see that no set of sequences can be broken up into subsets with these properties; and that no set has internal symmetries such as

$$S_{i,j} = r - S_{i,k}$$

for all j (see equation (2)) and hence deduce that the representations are irreducible. If we relax the upper bound $(\tilde{r}-1)$ on sequence entries then we obtain further representations by continuity (since there exist representations with irrational r arbitrarily close to any rational r). Some of these are reducible due to the vanishing of elements from equation (2). The remaining irreducible representations have apparent divergences caused by the vanishing of the denominator in equation (2). These divergences may be regulated by the prescription $r \to r + \varepsilon$, $1/\varepsilon$ contributions cancel after some ε-dependent similarity transformations and ε may then be set to zero. In what follows we restrict attention to cases in which the upper bound is present. The set of rational r values with a given $\tilde{r}$ correspond to the various branches associated with the Beraha q values (see Martin 1987) and the irreducible representations may be identified accordingly. Hereafter it will thus be sufficient to restrict attention to integer r (so $r = \tilde{r}$). Elsewhere (for irrational or imaginary r) we note that our representations coincide with the complete set of conventional irreducible representations found by Temperley (1986) using Young tableaux. The relevant tableaux are those with two rows of length $n+m$ and $n-m$.

Thus, while noting the existence of parasitic representations such as

$$U_1 = \mathrm{diag}(q^{1/2}, 0, 0, 0)$$
$$U_2 = q^{-1/2}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$
$$U_3 = \mathrm{diag}(0, q^{1/2}, 0, 0)$$

for $q = 2$ (from Martin 1986a) in which a conventional irreducible representation (the upper left 2×2 block) is host to other elements which do not form a representation on their own, it seems likely that we have found all the conventional representations. Note, for instance, that the representation obtained from sequences with $S_{1,j} = S_{2n+1,j} =$

2 is the same as that with $S_{1,j}=S_{2(n+1)+1,j}=1$ but with two fewer operators and is therefore reducible (cf equation (2)).

It is also easy to see from equation (2) that $m=0$ gives the only representation with $R_0 \neq 0$, where

$$R_0 = \prod_{\text{odd } i} U_i. \tag{4}$$

This means that the $m=0$ representation is responsible for the partition function in the associated statistical mechanical models (see Baxter 1982). For any product of U operators, χ, we have

$$R_0 \chi R_0 = \zeta_0(\chi) R_0 \tag{5}$$

where ζ_0 is a scalar. From equation (2) we see that objects of the form

$$R_m = \prod_{i=1}^{n-m} U_{2i-1} \prod_{j=n-m+1}^{n} (U_{2j-1} - q^{1/2})$$

are non-vanishing only in representation m, and may thus be used to form characteristic scalars $\zeta_m(\chi)$ in a way analogous to equation (5) (see also Temperley 1986). For example, the $r=4$, $n=2$, $m=1$ representation

$$U_1 = \begin{pmatrix} 2^{1/2} & \\ & 0 \end{pmatrix} \qquad U_2 = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad U_3 = \begin{pmatrix} 0 & \\ & 2^{1/2} \end{pmatrix}$$

has $R_0 = R_2 = 0$

$$R_1 = \begin{pmatrix} -2 & \\ & 0 \end{pmatrix}$$

and

$$U_1(U_3 - q^{1/2})\chi U_1(U_3 - q^{1/2}) = \zeta_1(\chi) U_1(U_3 - q^{1/2})$$

where ζ_1 is a scalar for any product χ by the relations (1). Furthermore, $R_t R_u = 0$ if $t \neq u$. Thus if χ is a statistical mechanical transfer matrix then these scalars will be associated with analytically disjoint parts of the transfer matrix spectrum (Martin 1986b).

We may obtain general expressions for the dimensions of our representations as follows. The dimension of a representation is given by the number, C, of sequences in the corresponding set. For a given n, r partition the set of C_n^r sequences into the $j=1, \ldots, r-2$ subsets of $C_{n,j}^r$ sequences which start

1 2 1 ...	$(j=1)$
1 2 3 2 ...	$(j=2)$
1 2 3 4 3 ...	$(j=3)$

etc, i.e. subsets of sequences whose kth entry is k $(k<j+2)$ and $(j+2)$th entry is j (note that for a given r only the first $r-2$ such subsets are allowed). The number of sequences in each subset, $C_{n,j}^r$, is $C_{n-1}^r - \Sigma_{j=1}^{r-4} C_{n-1,j}^r$ since C_{n-1}^r counts all possible endings from j at the $(j+2)$th position once, but overcounts by the number of sequences which do not get to j at the $(j+2)$th position starting from 1 at the third position (or equivalently those which do not get to j at the jth position in $2n-1$ element sequences, which is $\Sigma_{j=1}^{r-4} C_{n-1,j}^r$). In this way we obtain all the dimensions recursively from a knowledge of the first one:

$$C_n^r = \sum_{j=1}^{r-2} C_{n,j}^r. \tag{6}$$

Notice that this applies for all m, but that the first non-zero C_n^r is different in each case:

$$m=1 \qquad C_0^r=1 \qquad (r\geqslant 3)$$

$$m=2 \qquad C_1^r=1 \qquad (r\geqslant 4)$$

$$m=3 \qquad C_2^r=1 \qquad (r\geqslant 5)$$

etc.

The recursion may be written $C_n^r = C_n^{r-1}$ if $n+m<r-2$, and

$$\begin{aligned} C_n^r &= \sum_{j=1}^{[(r-1)/2]} \alpha_j^r C_{n-j}^r \\ &= (\alpha_1^{r-1}+1)C_{n-1}^r + \sum_{j=2}^{[(r-1)/2]} (\alpha_j^{r-1} - \alpha_{j-1}^{r-2})C_{n-j}^r \end{aligned} \tag{7}$$

where $[p]$ is the integer part of p, otherwise, whereupon the generating function for C_n^r with $m=0$ is as given in Martin (1987). That is, with

$$b_r(x) = \prod_{l=1}^{[(r-1)/2]} \left[1 - 4\cos^2\left(\frac{\pi l}{r}\right)x\right] \tag{8}$$

the generating function is

$$\begin{aligned} B_r^0(x) &\equiv 1 + \sum_{n=1}^{\infty} C_n^r x^n \\ &= \frac{b_{r-1}(x)}{b_r(x)}. \end{aligned} \tag{9}$$

Now with $m=1$, for instance, the generating function is

$$B_r^1(x) = \frac{(1-x)b_{r-2}(x)}{b_r(x)} \tag{10}$$

and so on. Notice that each representation has, up to overall factors, the same asymptotic dimensions for large n.

Now consider the trace, $\mathrm{tr}(q^{-1/2}U_i)$ (Martin 1987). Note that in any of our representations U_1 is diagonal with non-zero elements equal to $q^{1/2}$ in each of the C_{n-1}^r cases when the third entry in the corresponding basis sequence is 1. We can thus see that, at the Beraha q values, the large n limit representations all realise Jones' (1983) normalised trace condition.

Finally note that, if we remove the upper bound on integers in a sequence, then the $m=0$ representation becomes equivalent to the, in general reducible, Whitney polynomial representation described in Martin (1986a). Reducibility is manifested in the vanishing of certain elements given by equation (2). It is a trivial extension of the Whitney representation to include periodic boundary conditions. In general this extension is not so straightforward.

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